

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188		
<p>Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing this collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.</p>					
1. REPORT DATE (DD-MM-YYYY) August 2013		2. REPORT TYPE Journal Article		3. DATES COVERED (From - To) August 2013- January 2014	
4. TITLE AND SUBTITLE Development of a Multistage Reliability-Based Design Optimization Method			5a. CONTRACT NUMBER In-House		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Eric J. Paulson, Ryan P. Starkey			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER QOB7		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Air Force Research Laboratory (AFMC) AFRL/RQRC 10 E. Saturn Blvd. Edwards AFB, CA, 93524-7680			8. PERFORMING ORGANIZATION REPORT NO.		
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) Air Force Research Laboratory (AFMC) AFRL/RQR 5 Pollux Dr. Edwards AFB, CA, 93524-7048			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-RQ-ED-JA-2013-209		
12. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES Journal article published in the Journal of Mechanical Design Vol. 136, Issue #1 January 2013. PA Case Number: 13468; Clearance Date: 9 Sep 13. © 2014 Journal Article of Mechanical Design The U.S. Government is joint author of the work and has the right to use, modify, reproduce, release, perform, display, or disclose the work.					
14. ABSTRACT Complex system acquisition and its associated technology development have a troubled recent history. The modern acquisition timeline consists of conceptual, preliminary, and detailed design followed by system test and production. The evolving nature of the estimates of system performance, cost, and schedule during this extended process may be a significant contribution to recent issues. The recently proposed multistage reliabilitybased design optimization (MSRBDO) method promises improvements over reliabilitybased design optimization (RBDO) in achieved objective function value. In addition, its problem formulation more closely resembles the evolutionary nature of epistemic design uncertainties inherent in system design during early system acquisition. Our goal is to establish the modeling basis necessary for applying this new method to the engineering of early conceptual/preliminary design. We present corrections in the derivation and solutions to the single numerical example problem published by the original authors, Nam and Mavris, and examine the error introduced under the reduced-order reliability sampling used in the original publication. MSRBDO improvements over the RBDO solution of 10–36% for the objective function after first-stage optimization are shown for the original second-stage example problem. A larger 26–40% improvement over the RBDO solution is shown when an alternative comparison method is used than in the original. The specific implications of extending the method to arbitrary m-stage problems are presented, together with a solution for a three-stage numerical example. Several approaches are demonstrated to mitigate the computational cost increase of MSRBDO over RBDO, resulting in a net decrease in calculation time of 94% from an initial MSRBDO baseline algorithm.					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Eric Paulson
c. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NO (include area code) 661-525-5331

Development of a Multistage Reliability-Based Design Optimization Method

Eric J. Paulson

Air Force Research Laboratory,
Edwards AFB, CA 93524
e-mail: eric.paulson.1@us.af.mil

Ryan P. Starkey

University of Colorado,
Boulder, CO 80309
e-mail: rstarkey@colorado.edu

Complex system acquisition and its associated technology development have a troubled recent history. The modern acquisition timeline consists of conceptual, preliminary, and detailed design followed by system test and production. The evolving nature of the estimates of system performance, cost, and schedule during this extended process may be a significant contribution to recent issues. The recently proposed multistage reliability-based design optimization (MSRBDO) method promises improvements over reliability-based design optimization (RBDO) in achieved objective function value. In addition, its problem formulation more closely resembles the evolutionary nature of epistemic design uncertainties inherent in system design during early system acquisition. Our goal is to establish the modeling basis necessary for applying this new method to the engineering of early conceptual/preliminary design. We present corrections in the derivation and solutions to the single numerical example problem published by the original authors, Nam and Mavris, and examine the error introduced under the reduced-order reliability sampling used in the original publication. MSRBDO improvements over the RBDO solution of 10–36% for the objective function after first-stage optimization are shown for the original second-stage example problem. A larger 26–40% improvement over the RBDO solution is shown when an alternative comparison method is used than in the original. The specific implications of extending the method to arbitrary m-stage problems are presented, together with a solution for a three-stage numerical example. Several approaches are demonstrated to mitigate the computational cost increase of MSRBDO over RBDO, resulting in a net decrease in calculation time of 94% from an initial MSRBDO baseline algorithm. [DOI: 10.1115/1.4025492]

1 Introduction

Until relatively recently, optimization processes and methods were predominately focused on maximizing a system's discrete parameters of performance, particularly for complex systems. However, in the modern systems acquisition era, the research and development of complex engineering designs emphasizes system optimization for high reliability, ease of maintainability, and reduced risk of cost/schedule growth. Research into optimization under uncertainty and RBDO holds the promise of facilitating these types of system optimizations.

Ensuring system level optimization of reliability-type system attributes requires the integration of advanced numerical methods into the design and analysis processes to improve both the results and the efficiency of these acquisition processes. For high-cost complex systems, which require years of development before reaching production, the incorporation of uncertainty methods into the design, analysis, and optimization processes promises significant payoffs. However, broad implementation of uncertainty methods poses unique challenges in the context of analysis of alternatives and technology development. In 2007–2008, Nam and Mavris [1,2] proposed a MSRBDO method to accommodate the multiple design decision stages and realizations of uncertainty encountered during the early acquisition phases of complex systems, such as aircraft, under decreasing uncertainty. A notional example of a single evolving technical performance measure under uncertainty is illustrated in Fig. 1. Each of the periods

ending in a milestone can be considered as a discretized decision stage. Figure 2 illustrates the design decision and uncertainty realization process in the RBDO and MSRBDO approaches. Nam and Mavris described the general nomenclature for MSRBDO and compared the results for a two-stage reliability-based design optimization (2SRBDO) numerical test problem to deterministic and RBDO formulation solutions. An extension to the nomenclature to accommodate the case where uncertainty variables/parameters change between interim stages while remaining in uncertainty parameter form expressed as a probability distribution, prior to final realization as a deterministic value, is described in Ref. [4].

There are errors in the original derivation and solution to the 2SRBDO test problem of Nam and Mavris. The primary source of the solution error was found to be the use of reduced-order Monte Carlo sampling (MCS). The original work combined Monte Carlo sampling over the first-stage uncertainty parameters with quantile approximations for the second-stage reliability terms. This paper

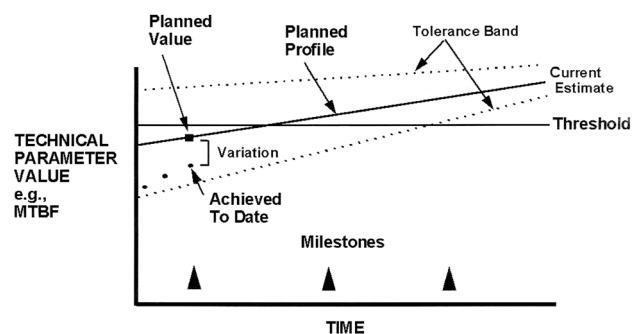


Fig. 1 System performance measures/uncertainties over acquisition timeline [3]

Contributed by the Design Automation Committee of ASME for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received March 29, 2013; final manuscript received August 23, 2013; published online November 7, 2013. Assoc. Editor: David Gorsich.

This material is declared a work of the US Government and is not subject to copyright protection in the United States. Approved for public release; distribution is unlimited; PA-13468.

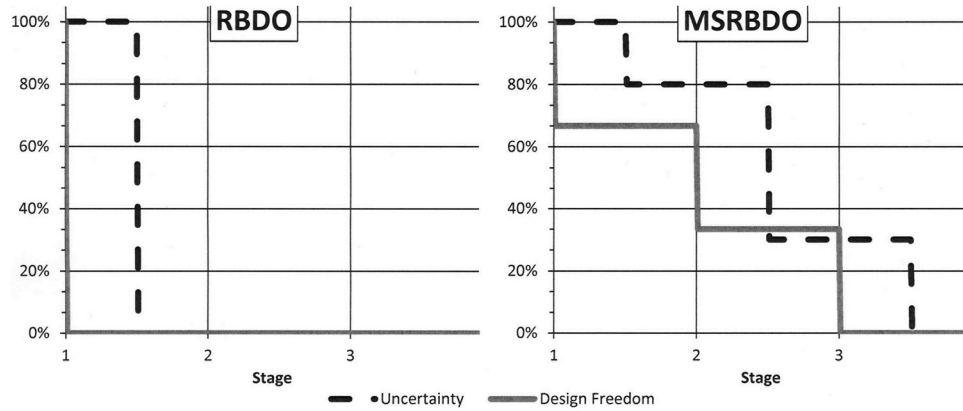


Fig. 2 Comparison of decision/realization in RBDO and MSRBDO

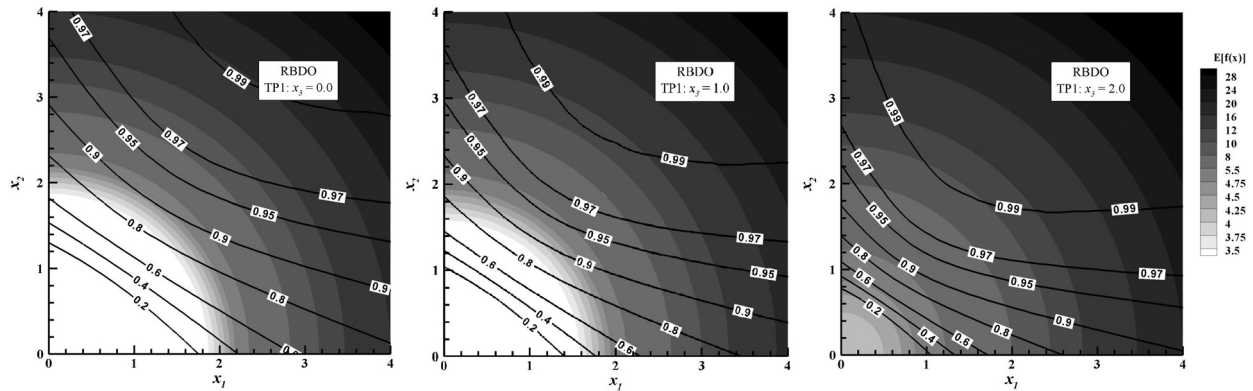


Fig. 3 Plots of RBDO solutions for $x_3 = (a) 0, (b) 1, \text{ and } (c) 2$

describes the both derivation errors and numerical errors introduced by reduced-order sampling in the original test problem and applies the method to more general multistage forms of problems. Some approaches are described, which reduce the sampling error while mitigating the computational costs of higher order sampling.

2 MSRBDO Results and Discussion

2.1 Formulation of 2SRBDO Test Problem 1 (Nam–Mavris)

2.1.1 Nam–Mavris Formulation. A minor change to the Nam and Mavris nomenclature was made. The design variables are represented here as x_1, x_2 , and x_3 rather than the original x, y , and z in order to generalize the problem form for arbitrary numbers of design variables.

It is illustrative to derive the 2SRBDO formulation starting from a deterministic form. An example of this deterministic problem form can be expressed as

$$\min_{\bar{x}} f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \quad (1)$$

$$\text{s.t. } \xi_1 x_1 + \xi_2 x_2 + x_3 - 5 \geq 0 \quad (2)$$

where $(x_1, x_2) \in [0, \infty)^2$, $x_3 \in [0, 2]$, and the bolded vector ξ represents the constraint function constants $\xi_1 = 2$ and $\xi_2 = 3$. This is a nonlinear problem with linear inequality constraints.

This is now reformulated in RBDO probabilistic constraint form, with ξ now representing the set of epistemic (reducible) uncertainty parameters, modeled as normal distributions $\xi_1 \sim \mathcal{N}(2, 1^2)$ and $\xi_2 \sim \mathcal{N}(3, 1^2)$. Again, the design variables are

bounded: $(x_1, x_2) \in [0, \infty)^2$ and $x_3 \in [0, 2]$. After the addition of a reliability target, α , to the constraint equation given in Eq. (2), the single-stage RBDO problem is now in the form shown in Eqs. (3)–(6). A reliability target is the minimum acceptable probability of satisfying the inequality constraint

$$\min_x f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \quad (3)$$

$$\text{s.t. } P[\xi_1 x_1 + \xi_2 x_2 + x_3 - 5 \geq 0] \geq \alpha \quad (4)$$

$$(x_1, x_2) \geq 0 \quad (5)$$

$$0 \geq x_3 \geq 2 \quad (6)$$

In general form, the probabilistic constraint for this type of problem can be expressed using Eq. (7), where nx is the number of design variables

$$P \left[\alpha_0 + \sum_{i=1}^{nx} (a_i \xi_i + b_i) x_i \geq 0 \right] \geq \alpha \quad (7)$$

Figures 3(a)–3(c) illustrates the solution space of interest for the RBDO and deterministic problems in the x_1 – x_2 plane, for slices at $x_3 = 0.0, 1.0$, and 2.0 , respectively. The objective function values are represented by the banded grayscale mapping, and the reliability of meeting the feasibility constraint equation is shown by the labeled black isolines. For this problem, the objective function values are identical for the deterministic and RBDO forms of the problem at any given (x_1, x_2, x_3) design point, because $f(x)$ is not an explicit or implicit function of ξ .

The single-stage RBDO problem can now reformulated into two sequential design decision stages, with the design variables partitioned into a first-stage decision, nominally $x^{(1)} = (x_1, x_2)$,

and a second-stage decision, $x^{(2)} = (x_3)$. Similarly, the uncertainty parameters can be partitioned into a first-stage uncertain parameter, $\xi^{(1)}$, and a second-stage parameter, $\xi^{(2)}$. We will use the problem form $\xi^{(1)} = \xi_1$ and $\xi^{(2)} = \xi_2$, for the purposes of the derivations that follow. However, solutions for each of the two alternative formulations will be presented. The reliability target is similarly split into stage reliability targets at each optimization stage, $\alpha = (\alpha^{(1)}, \alpha^{(2)})$. In general, the reliability target at each subsequent stage should increase, representing an increasing likelihood of meeting the problem constraints as design decisions are made.

This now sets up two sequential optimization problem stages. The first decision stage selects the optimal members of $x^{(1)}$ s.t. $P_{\xi^{(1)}}[g \geq 0] \geq \alpha^{(1)}$, which is followed by a realization of the value for the first-stage uncertainty parameter $\xi^{(1)} \rightarrow \hat{\xi}^{(1)}$. In the second-stage problem, the optimal second-stage decision variable x_3 is then selected s.t. $P_{\xi^{(2)}}[g \geq 0] \geq \alpha^{(2)}$, determining the final value of the objective function. The realization of the final uncertainty parameters in set $\xi^{(2)}$ then determines whether the constraint function inequality $g \geq 0$ was met. The first-stage optimization is expressed mathematically by Eqs. (5) and (6) and by

$$\min_{x^{(1)}} E[f(x^{(1)}, x^{(2)*}, \xi^{(1)}, \xi^{(2)})] = x_1^2 + x_2^2 + E[x_3^2] \quad (8)$$

$$\text{s.t. } P[\xi_1 x_1 + \xi_2 x_2 + x_3 - 5 \geq 0] \geq \alpha^{(1)} \quad (9)$$

For the specific case of $\xi^{(1)} = \xi_1$ and $\xi^{(2)} = \xi_2$, the second-stage optimization is now expressed by Eq. (6) and by

$$\min_{x^{(2)}} E[f(\hat{x}^{(1)}, x^{(2)}, \hat{\xi}^{(1)}, \xi^{(2)})] = \hat{x}_1^2 + \hat{x}_2^2 + x_3^2 \quad (10)$$

$$\text{s.t. } P[\hat{\xi}_1 \hat{x}_1 + \xi_2 \hat{x}_2 + x_3 - 5 \geq 0] \geq \alpha^{(2)} \quad (11)$$

Recognize that the second-stage design variable, x_3 , will be determined after the realization of the first-stage uncertainty parameter, $\xi_1 \rightarrow \hat{\xi}_1$, since the probabilistic constraint equation is now in deterministically equivalent form. Therefore, during the first-stage problem, the expected optimal value of x_3 , written as x_3^* , is a function of $\hat{\xi}_1, \hat{x}_1, \hat{x}_2$. So, x_3^* now behaves as an implicit uncertainty parameter during the first-stage optimization. This in turn makes the first-stage objective function an implicit function of the uncertainty parameters as well as an explicit function of the first-stage design variables. For this specific problem, an increase in x_3^* value increases the feasibility of the constraint (desired); however, it also increases the numerical value of the objective function (not desired). Due to this relationship, if x_3 were unbounded, x_3^* would be determined such that $P_{\xi^{(2)}|\xi^{(1)}}[g \geq 0] = \alpha^{(2)}$, as stated by Nam and Mavris. Their first-stage optimization calculated objective value and reliability expectations by reduced-order MCS over only $\xi^{(1)}$ with $N_1 = 10^6$. The value of $\xi^{(2)}$ was replaced in Ref. [2] with the second-stage reliability target quantile value $\xi_2 = \Phi_{\xi_2}^{-1}(\alpha^{(2)})$. The corrected formulation is provided below.

2.1.2 Paulson–Starkey Formulation. Algebraically solving the constraint function from Eq. (11) for x_3^* at the limit state $g = 0$ yields Eq. (12) after the substitution $\xi_2 = \Phi_{\xi_2}^{-1}(\alpha^{(2)})$

$$x_3^* = -\hat{x}_1 \hat{\xi}_1 - \hat{x}_2 \Phi_{\xi_2}^{-1}(1 - \alpha^{(2)}) + 5 \quad (12)$$

This equation for x_3^* differs from the original solution given in Nam and Mavris [2] by both the sign of the second term on the RHS of Eq. (12) as well as the argument for inverse cumulative distribution function, $\Phi_{\xi_2}^{-1}(\cdot)$. In addition, for this problem, x_3 is indeed a bounded variable, with lower and upper limits x_{3LL} and x_{3UL} , respectively. Therefore, solutions of Eq. (12) for particular combinations of x and $\xi^{(1)}$ may result in values for x_3^* , which fall outside the bounds on the design variable x_3 . Therefore, calculating the exact reliability requires sampling over both $\xi^{(1)}$ and $\xi^{(2)}$, due to the necessity of a filter step applying x_3 bounds after the algorithm calculates x_3^* . Although Nam and Mavris [2] discuss the effect of bounding on x_3 , MCS was performed only over $\xi^{(1)}$. This introduces an error in the estimated reliability at stage 2, after the first-stage optimization.

After making the corrections to the original x_3^* derivation, a first-stage optimization dual-loop MCS algorithm was written for an exhaustive grid-pattern search over the design variable domain with fixed spatial step sizes of $dx = \pm 0.0005$. This is assumed to be of the order of the solution by Nam–Mavris based on the number of significant digits in those published results. The optimization loop did not utilize any gradient information for the objective function or constraint reliability. The outer loop calculates the expected value of the objective function $E[f(x)]$ at each (x_1, x_2) design point by sampling only over the first-stage uncertainty, which is referred to as first-order objective function sampling (FOOFS). Future stage uncertainty parameters would be replaced by the value from the inverse CDF function. If one wanted to sample over first- and second-stage uncertainties, this is referred to as SOOFS for second order. In the reliability loop, sampling over first-stage uncertainty to calculate the reliability as utilized in Ref. [2] is referred to as first-order reliability sampling (FORS). Sampling over first- and second-stage uncertainty would be second-order reliability sampling (SORS), and for arbitrary m , first to third would be 3ORS, etc. In general, sampling over anything less than the full m stages of a problem is referred to here as reduced-order sampling.

To calculate solutions for the 2SRBDO problem, we evaluated alternative MCS formulations for the reliability calculation, which are summarized in Table 1. It is presumed that Nam and Mavris used the formulation titled FORS-II. Our solution utilizes the SORS-I formulation. To evaluate the impact of the corrections in the derivation of Eq. (12) and use of SORS, $E[f(x)]$ and $P[g(x, \xi) \geq 0]$ were calculated for $N_1 = N_2 = 10^6$, at the location of the published first-stage optima from the original solution in Ref. [2]. A comparison of these results is shown in Table 2. The Nam paper did not publish the values for $E[x_3^*]$ at their optima; so, no comparison could be made of those values. Although the differences in the objective function values are fairly minor with relative errors of less than 0.2%, the reliabilities for the constraint function at those $x^{(1)}$ values failed to meet the problem's first-

Table 1 Reliability sampling formulations

In $g(x, \xi)$:	$\xi_2 =$	$x_3 =$	Comments
FORS-I	$\phi_{\xi_2}^{-1}(1 - \alpha^{(2)})$	$E[x_3^*]$	For comparison only
FORS-II	$\phi_{\xi_2}^{-1}(1 - \alpha^{(2)})$	x_3^* from Eq. (12), bound by x_{3LL} and x_{3UL}	ξ_2, x_3 terms retain dependence on $\alpha^{(2)}$
FORS-III	$\phi_{\xi_2}^{-1}(1 - \alpha^{(2)})$	x_3^* from Eq. (12), not bound by x_{3LL} or x_{3UL}	$g = 0$ regardless of $\hat{\xi}_1$ value
SORS-I	$\hat{\xi}_2$	x_3^* from Eq. (12), bound by x_{3LL} and x_{3UL}	x_3 term retains dependence on $\alpha^{(2)}$
SORS-II	$\hat{\xi}_2$	$5 - \hat{\xi}_1 \hat{x}_1 - \hat{\xi}_2 \hat{x}_2$	No terms retain $\alpha^{(2)}$ dependence

Table 2 Discrepancy with published results

	$\alpha^{(1)} = 0.95$				$\alpha^{(1)} = 0.97$				$\alpha^{(1)} = 0.99$			
	$\alpha_{TS}(\xi_1)$ Nam	$\alpha_{TS}(\xi_1)$ Paulson	$\alpha_{TS}(\xi_2)$ Nam	$\alpha_{TS}(\xi_2)$ Paulson	$\alpha_{TS}(\xi_1)$ Nam	$\alpha_{TS}(\xi_1)$ Paulson	$\alpha_{TS}(\xi_2)$ Nam	$\alpha_{TS}(\xi_2)$ Paulson	$\alpha_{TS}(\xi_1)$ Nam	$\alpha_{TS}(\xi_1)$ Paulson	$\alpha_{TS}(\xi_2)$ Nam	$\alpha_{TS}(\xi_2)$ Paulson
x_1	1.276	1.2760	0.659	0.6590	1.397	1.3970	0.736	0.7360	1.611	1.6110	1.205	1.2050
x_2	1.218	1.2180	1.585	1.5850	1.322	1.3220	1.726	1.7260	1.993	1.9930	2.168	2.1680
$E[f(x)]$	4.807	4.816	3.901	3.906	5.665	5.675	4.469	4.475	8.176	8.188	6.818	6.820
$P[g \geq 0]$	0.95	0.899	0.95	0.838	0.97	0.936	0.97	0.885	0.99	0.978	0.99	0.960

```

SET (  $\hat{x}_1, \hat{x}_2$  )
RESET Mean_ $x_3$ , Mean_ $x_3sq$ , Mean_ $E[f(x)]$ 
SAMPLE  $\xi_1$ 
 $x_3^* =$  _____
LIMIT  $x_{3LL} \leq x_3^* \leq x_{3UL}$ 
Mean_ $x_3 =$  Mean_ $x_3 + x_3^*$ 
Mean_ $x_3sq =$  Mean_ $x_3sq + (x_3^*)^2$ 
NEXT  $\xi_1$ 
Mean_ $x_3(\hat{x}_1, \hat{x}_2) =$  Mean_ $x_3 / N_1$ 
Mean_ $x_3sq(\hat{x}_1, \hat{x}_2) =$  Mean_ $x_3sq / N_1$ 
 $E[f(\hat{x}_1, \hat{x}_2)] = (\hat{x}_1)^2 + (\hat{x}_2)^2 +$  Mean_ $x_3sq(\hat{x}_1, \hat{x}_2)$ 
IF  $\{E[f(\hat{x}_1, \hat{x}_2)] < E[f(x)]_{best}\}$ , GOTO Reliability Loop
NEXT (  $\hat{x}_1, \hat{x}_2$  )

```

Fig. 4 FOOFS objective function loop logic

```

ENTER from FOOFs Reliability Loop
Fcount = 0
SAMPLE  $\xi_1$ 
SAMPLE  $\xi_2$ 
 $x_3 =$  _____
LIMIT  $x_{3LL} \leq x_3 \leq x_{3UL}$ 
 $g = \xi_1 * \hat{x}_1 + \xi_2 * \hat{x}_2 + x_3 - 5$ 
IF  $\{g < 0\}$ , Fcount = Fcount + 1
NEXT  $\xi_2$ 
NEXT  $\xi_1$ 
 $P[g \geq 0] = 1 - (Fcount / (N_1 * N_2))$ 
RETURN

```

Fig. 6 SORS reliability loop logic

```

ENTER from FOOFs Reliability Loop
Fcount = 0
SAMPLE  $\xi_1$ 
 $g = \xi_1 * \hat{x}_1 + \xi_2 * \hat{x}_2 + x_3 - 5$ 
IF  $\{g < 0\}$ , Fcount = Fcount + 1
NEXT  $\xi_1$ 
 $P[g \geq 0] = 1 - (Fcount / N_1)$ 
RETURN

```

Fig. 5 FORS reliability loop logic

stage target reliability for all $\alpha^{(1)}$ cases. The shortfalls in the calculated reliabilities from the reliability target ranged from 1.19% to 11.18%. It is unclear from Ref. [2] whether published values for $P_{\xi}[g(x, \xi) \geq 0]$ were actually calculated via FORS-II or if they were assumed, since these results were shown with a different number of significant digits than the other published results. Pseudocode for the FOOFs, FORS, and SORS logic implemented here is shown in Figs. 4–6, respectively.

The first-stage optimization for the test problem was performed using the FOOFs/SORS problem formulation. The solution space of the two alternative first-stage optimization formulations for the 2SRBDO is changed fundamentally from the original RBDO solution. It was illustrative to again map the first-stage solution space of this 2SRBDO problem. Again the objective function is shown in grayscale contours, and the reliability values for the constraint with labeled isolines, Fig. 7 shows the results for first-stage optimization of the 2SRBDO for the case $\xi^{(1)} = \xi_1$, and Fig. 8 for the 2SRBDO case $\xi^{(1)} = \xi_2$, for reliability target values of $\alpha^{(1)} = 0.95$ and $\alpha^{(2)} = 0.97$. Contrast this with the plot of Fig. 3(b) for the RBDO solution at $x_3 = 1.0$, which is near the RBDO problem's optimal value of $x_3 = 1.03$.

Reformulating an RBDO problem in MSRBDO form clearly changes the nature of the optimization problem. The formerly deterministic objective function is now an implicit function of ξ . It follows from Figs. 7 and 8 that, if an engineering problem can be represented as a MSRBDO problem and if decision maker

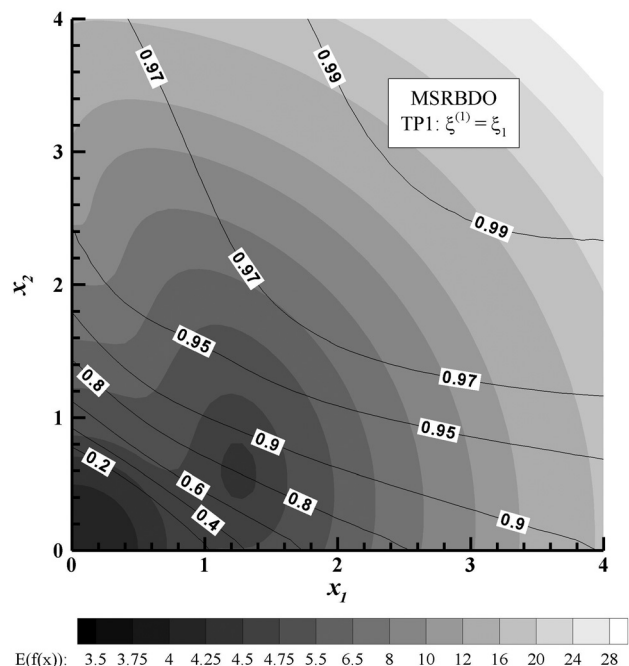


Fig. 7 2SRBDO test problem first-stage objective function and reliability values for ξ_1

discretion exists in determining which uncertainty parameter(s) to realize (reduce) earlier, then a quantifiable improvement to the optimal objective function value can be used as the quantitative decision criteria for that determination.

The computational burden was significantly increased by sampling over both $\xi^{(1)}$ and $\xi^{(2)}$ during the reliability loop. In the limit, computation time is proportional to N^m . As a result, a compromise in sampling resolution was utilized in the solution of the test problem. Since it would be possible to correct errors due to sampling size in the second-stage uncertainty parameters during

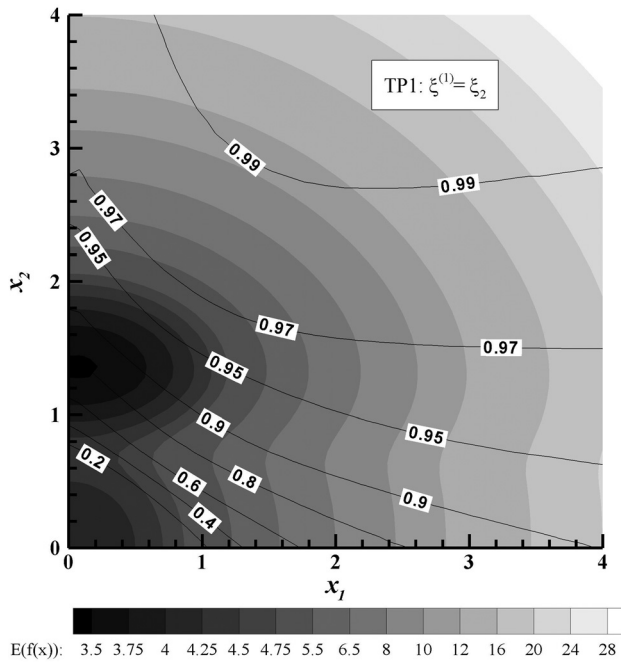


Fig. 8 2SRBDO test problem first-stage objective function and reliability values for ξ_2

that later stage of optimization, the first- and second-stage sampling sizes were set differently at $N_1 = 10^4$ and $N_2 = 10^3$. Second, in the outer optimization loop, at each grid point the minimum possible value for the objective function was calculated using $x_3 = x_{3LL}$. If this minimum value would not result in replacing the current optimal point, calculating the reliability of this point was unnecessary and the algorithm moved on to the next grid point. This resulted in reduced computation times at each spatial resolution setting. The final optimization results for each of the original cases are shown in Table 3. In the table, x_{DET} represents the deterministic problem solutions, x_{SS} denotes the single-stage RBDO problem solutions, and x_{TS} refers to the two-stage MSRBDO problem solutions.

The table shows that, in every feasibility target case, the first-stage $E[f(x)]$ resulting from the 2SRBDO method is more optimal (lower) than the solution to the single-stage RBDO formulation. Therefore, the original justification for the MSRBDO method still holds. By reformulating the RBDO problem over multiple decision/realization steps, a more optimal objective function value results for the same value of the reliability target. Final values of the expected objective function improved 9.8–36% by using the MSRBDO method in lieu of RBDO for this example. Additionally, there exists a preferred order for realizing the uncertainty parameters in a multistage RBDO problem. For this problem, using $\xi^{(1)} = \xi_2$ resulted in a more optimal $E[f(x)]$ at the first-decision stage.

Table 4 RBDO–2SRBDO comparison by final α

$\alpha^{(1)} = 0.95, \alpha^{(2)} = 0.97$					
	$x_{SS} \alpha = 0.97$	$x_{TS} \xi^{(1)} = \xi_1$	% Change	$x_{TS} \xi^{(1)} = \xi_2$	% Change
x_1	1.2169	1.2670	4.1	0.5059	−58.4
x_2	1.7975	1.1062	−38.5	1.5806	−12.1
$x_3, (E[x_3^*])$	1.2631	(1.1101)	−12.1	(0.6434)	−49.1
$E[f(x)]$	6.3073	4.6640	−26.1	3.7462	−40.6

As a final observation of the results of Nam–Mavris, comparing MSRBDO to RBDO solutions, from an engineering perspective, it is more consistent to compare values of the objective functions for identical values of the final constraint reliabilities. In other words, results for single-stage RBDO with $\alpha = 0.97$ would be more appropriately compared with the results for 2SRBDO results for $\alpha^{(2)} = 0.97$, since that is the requirement for the reliability of satisfying the constraint after optimizing all the available design variables. Table 4 compares the RBDO and 2SRBDO problem results in this manner for the case of $\alpha^{(2)} = 0.97$. Objective function improvements of 26.1% and 40.6% show the MSRBDO formulation to be even more advantageous than the RBDO form when compared thusly.

2.1.3 Computational Efficiency. As mentioned previously, the MSRBDO MCS algorithm was experimented with in order to reduce the computational overhead that accompanies Monte Carlo sampling over more than the first-stage uncertainty. A comparison was made between 2SRBDO solutions to the same problem above for a 101 by 101 x_1 – x_2 grid with $dx_1 = dx_2 = 0.015$ for three different solution strategies: baseline solution sampled at $N_1 = N_2 = 10^4$, the pre-SORS design point elimination solution and $N_1 = N_2 = 10^4$, and a solution implementing both pre-SORS design point elimination and reduced second-stage sampling with $N_1 = 10^4$ and $N_2 = 10^3$. The resulting calculation times and objective function values are shown in Fig. 9. The pre-SORS point elimination alone reduced computational time by 33.9%, and using both pre-SORS point elimination along with reduced sampling for second-stage uncertainty reduced solution time 94.3%.

Since the outer optimization loop stepped through decreasing dx step sizes by 50% until the solution's spatial discretization criterion was met, a series of runs were performed to evaluate any the change in effectiveness of the pre-SORS design point elimination as the algorithm neared the final optima for the $\xi^{(1)} = \xi_1$ problem with all three cases of $\alpha^{(1)} = 0.95, 0.97, 0.99$. The fraction of design points that could be eliminated before the SORS loop decreased as the algorithm neared the optima and increased as the problem's $\alpha^{(1)}$ increased. These results are shown in Fig. 10. Therefore, pre-SORS point elimination is most effective during the first three to four resolution steps.

2.2 3SRBDO Test Problem. A three-stage MSRBDO test problem was investigated to identify general characteristics of problems solvable with a MSRBDO method for arbitrary numbers of stages, m . The single-stage RBDO problem is reformulated into

Table 3 2SRBDO results using FOOFs/SORS

	$\alpha^{(1)} = 0.95, \alpha^{(2)} = 0.97$				$\alpha^{(1)} = 0.97, \alpha^{(2)} = 0.99$			$\alpha^{(1)} = 0.99, \alpha^{(2)} = 0.995$		
	x_{DET}	x_{SS}	$x_{TS} \xi^{(1)} = \xi_1$	$x_{TS} \xi^{(1)} = \xi_2$	x_{SS}	$x_{TS} \xi^{(1)} = \xi_1$	$x_{TS} \xi^{(1)} = \xi_2$	x_{SS}	$x_{TS} \xi^{(1)} = \xi_1$	$x_{TS} \xi^{(1)} = \xi_2$
x_1	0.7140	1.1030	1.2670	0.5059	1.2169	1.3531	0.5437	1.3335	1.4250	0.9695
x_2	1.0715	1.7010	1.1062	1.5806	1.7975	1.2381	1.6931	2.0245	1.8759	2.1172
$x_3, (E[x_3^*])$	0.3755	1.0300	(1.1101)	(0.6434)	1.2631	(1.2247)	(0.6226)	1.9040	(1.1620)	(0.4132)
$E[f(x)]$	1.7857	5.1709	4.6640	3.7462	6.3073	5.4622	4.1365	9.5020	7.5324	6.0668
$P[g \geq 0]$	0.5000	0.9500	0.9500	0.9500	0.9700	0.9700	0.9700	0.9900	0.9900	0.9900

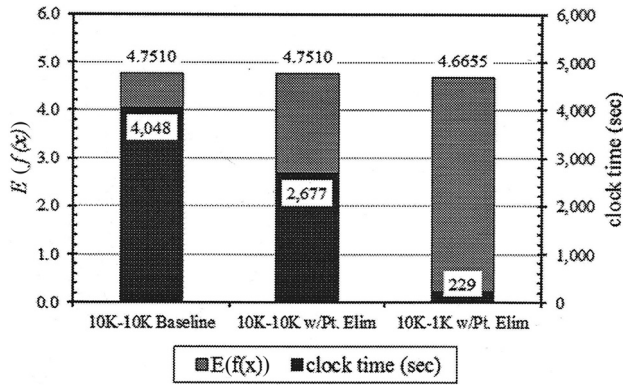


Fig. 9 Test problem 1 code speed improvement results

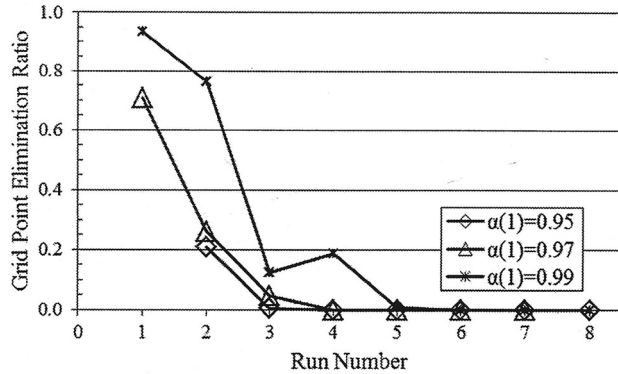


Fig. 10 Decreasing effectiveness of pre-SORS point elimination near region of optima

three sequential design decision stages, with the design variables partitioned into a first-stage decision $x^{(1)} = (x_1)$, a second-stage decision $x^{(2)} = (x_2)$, and a third-stage decision $x^{(3)} = (x_3)$. The design variables are bounded, with $x_1 \in [0, \infty)$, $x_2 \in [0.5, 4.5]$, and $x_3 \in [0.5, 2.5]$. The stage uncertainty parameters were the Gaussian distributions $\xi^{(1)} = \xi_1 \sim N(2, 0.3^2)$, $\xi^{(2)} = \xi_2 \sim N(1, 0.3^2)$, and $\xi^{(3)} = \xi_3 \sim N(3, 0.3^2)$. The stage reliability targets were set at $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}) = (0.7, 0.8, 0.9)$ for both inequality constraints. This sets up three sequential optimization problems. The first decision stage selects the optimal member of $x^{(1)}$ s.t. $P_{\xi^{(1)}}[g \geq 0] \geq \alpha^{(1)}$, which is followed by a realization of the value for uncertainty parameter $\xi^{(1)} \rightarrow \hat{\xi}^{(1)}$. The first-stage optimization is now given by

$$\min_{x^{(1)}} E[f(x^{(1)}, x^{(2)*}, x^{(3)*}, \xi^{(1)}, \xi^{(2)}, \xi^{(3)})] = x_1^2 + x_2^2 + x_3^2 \quad (13)$$

$$\text{s.t. } P[g_1 : \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 - 10 \geq 0] \geq \alpha^{(1)} \quad (14)$$

$$P[g_2 : 0.5\xi_1 x_1 + 0.5\xi_2 x_2 - 7 \geq 0] \geq \alpha^{(1)} \quad (15)$$

For this particular form of MSRBDO problem, at the r th decision stage, the j th constraint equation can be expressed in the general form of Eq. (16), where again nx is the number of design variables

$$P\left[a_{0j} + \sum_{i=1}^{nx} a_{ij}\xi_i x_i \geq 0\right] \geq \alpha^{(r)} \quad (16)$$

For this form of MSRBDO inequality constraint system of equations, it can be shown using the fundamental theorem for linear systems [5] that unique solutions at the limit state $g = 0$ for the

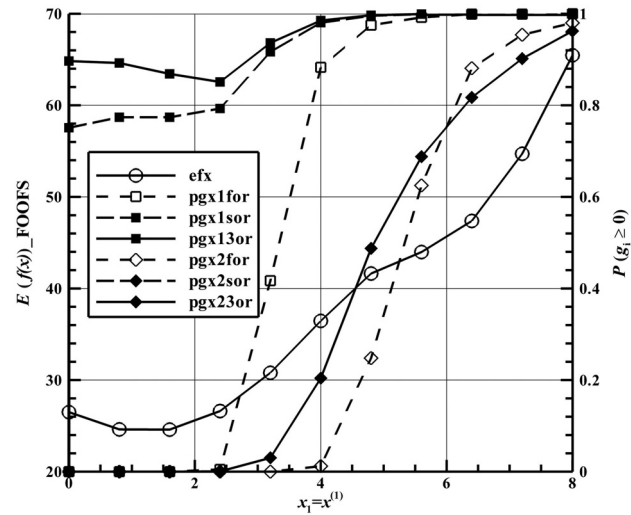


Fig. 11 3SRBDO test problem objective function and reliability vs. x_3

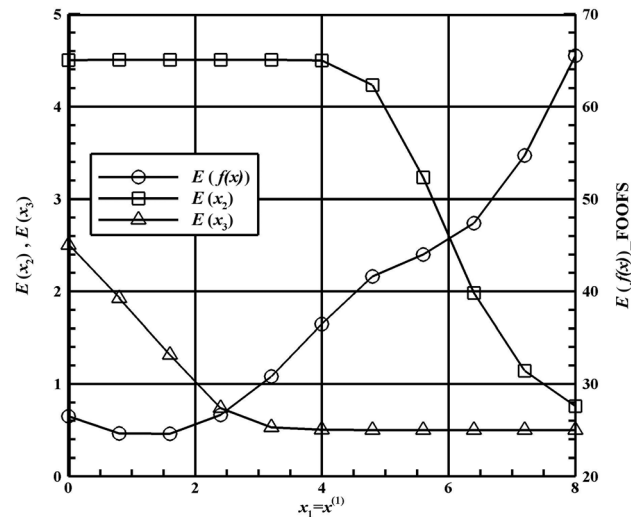


Fig. 12 $E[x_2]$, $E[x_3]$, and $E[f(x)]$ vs. x_1

future stage design variables, $x^{(j>r)*}$, exist at each first-stage design point $\hat{x}^{(1)}$ and first-stage uncertainty realization under consideration only if $n_c \geq m - 1$.

The results of first-stage problem are plotted for $0 \leq x_1 \leq 8$ in Fig. 11 at a spatial resolution of $dx = 0.8$. To illustrate the error introduced via RORS sampling, the reliabilities of g_1 and g_2 were calculated via first-order, second-order, and third-order reliability sampling and are shown on the second y-axis of the figure. The objective function is plotted with open circles on the primary y-axis. Since the first-stage reliability target is 0.70, it is clear that g_1 , shown in square symbols, would be treated as an active constraint only if FORS (shown by dashed line with unfilled symbols) was used to calculate $P[g_1(x) \geq 0]$ (shown in the legend as pgx1for). Use of SORS or 3ORS correctly indicates that only g_2 is an active constraint. Since the g_2 constraint is only a function of ξ_1 and ξ_2 , there is no dependency on third-stage uncertainty. Therefore, the SORS and 3ORS solutions for meeting the g_2 constraint are identical and the curves for pgx2sor and pgx23or in Fig. 11 are coincident.

The expectation values of x_2 , x_3 , and $f(x)$ vs. x_1 are shown in Fig. 12, indicating where the upper and lower limits on x_2 and x_3 are encountered.

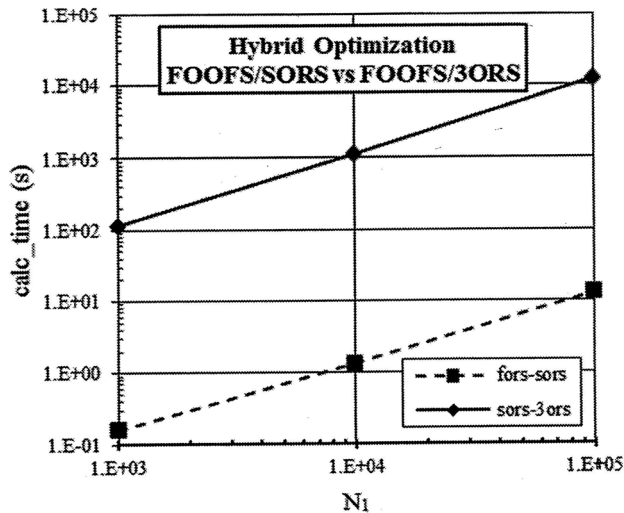


Fig. 13 Hybrid gradient optimization computational costs

2.3 Hybrid Reduced-Order Reliability/Full Reliability Optimization Algorithm. Full-order reliability sampling over every stage's uncertainty parameters imposes a high computational cost for anything but the simplest optimization MSRBDO problems, particularly when the outer optimization loop is a simple grid marching algorithm. A hybrid algorithm, which performs spatial grid marching over a bounded region using reduced-order reliability sampling to determine an initial global optimum, was combined with a second correction phase using a gradient-based algorithm employing full-order reliability sampling to solve iteratively for $E[f(x)]$ until meeting the condition $|\alpha^{(i)} - P[g(x) \geq 0]| \leq \alpha_{\text{criterion}}$. The Newton-Raphson root-finding method was selected for this correction phase since it only requires the first derivative of the function, and the method can be generalized in multiple dimensions. In addition, near the root, the method converges quadratically, with the number of significant digits of the solution nearly doubling per iteration [6]. The algorithm was demonstrated for the 3SRBDO problem above using a FOFS/FORS phase one and FOFS/SORS during the correction phase. To begin the correction phase, the last optima from phase one at $x_1 = 6.0$ has its SORS reliability calculated. The initial gradient in reliability at $x_1 = 6.0$ was calculated using first-order backward difference with the SORS reliability calculated for the point $x_1 = 6.0 - dx = 5.4$, from the previous grid marching phase. For subsequent iterations, the algorithm only requires calculation of SORS reliability for the next x . For $\alpha_{\text{criterion}} = 0.005$, the optima was found after one step. More generally, if g_1 had been the active constraint in this problem, the desired level of accuracy may have required the user to utilize a FOFS/3ORS during the second correction phase. Figure 13 shows the difference in computational time for FORS/SORS vs. SORS/3ORS at three different sampling levels for $N_1 = (10^3, 10^4, 10^5)$. In each of the three cases, $N_2 = N_3 = 10^3$.

3 Summary

A MSRBDO Monte Carlo sampling solution algorithm has been demonstrated using a dual-loop optimization/reliability sampling approach for two numerical examples of probabilistically constrained problems with linear inequality constraints. Derivation and solution errors in the originally published solution of the two-stage MSRBDO problem were corrected, and the origin of the error was shown to stem from use of reduced-order reliability sampling when future-stage design variables are bounded. Improvements to the objective function of 10–40% were obtained when a MSRBDO problem formulation is utilized over that of

RBDO. Algorithm modifications were demonstrated to significantly reduce the computational cost of a MSRBDO full-order reliability sampling formulation, resulting in a net clock-time reduction of 94%. The two improvements demonstrated first were (1) the use of different sample sizes per stage and (2) grid point elimination before reliability sampling. A third improvement using a hybrid reduced-order reliability nongradient phase/full-order reliability gradient phase algorithm was then successfully demonstrated to reduce the cost of higher order reliability sampling, while retaining that higher order accuracy near the final solution.

More generally, a consistent nomenclature has been proposed here for future discussions of m -stage MSRBDO results. This nomenclature insures a clear description of the levels of reduced-order uncertainty sampling utilized. Finally, an alternative approach for comparing RBDO to MSRBDO results is proposed, where results are compared for $\alpha_{\text{RBDO}} = \alpha_{\text{MSRBDO}}^{(m)}$, which apply more consistently to realistic engineering applications.

4 Future Work

An application of the MSRBDO method to an expendable launch vehicle engineering design optimization problem is ongoing. In addition, the authors intend to identify and implement additional modifications to the existing algorithm to increase the optimization loop convergence efficiency and further reduce the computational costs of the reliability calculation, which are a drawback for the MSRBDO formulation. Additional work in this area is required to extend application of the numerical method to a broader range of engineering problems. Specific extensions under consideration include: application to systems of nonlinear constraint equations, problems with multiple uncertainty parameters per stage, and more complex forms of the uncertainty parameters in the probabilistic constraint equations such as $a_i x_i / \xi_i$ or $a_i \xi_i \xi_j x_i$. Finally, more advanced optimization algorithms should be tested for the initial optimization phase, which can accommodate the presence of multiple local optima. The DIRECT algorithm is one such algorithm, which being considered for local optima tolerance [7].

Nomenclature

- $E[\cdot]$ = expectation operator
- N_j = number of samples of j th stage uncertainty parameters
- $\mathcal{N}(\mu, \sigma^2)$ = normal distribution of mean μ and variance σ^2
- $P[\cdot]$ = probability
- f = objective function
- g_i = i th constraint function
- m = number of decision stages in the problem
- n_c = number of constraint equations
- x = set of deterministic decision variables
- x_i = i th member of decision variable set
- $x^{(j)}$ = decision (design) variables determined during j th stage
- $\hat{x}^{(j)}$ = realizations of j th stage decision variables
- $x^{(j)*}$ = expected values for j th stage decision variables at stages prior to j
- α = RBDO feasibility target
- $\alpha^{(j)}$ = MSRBDO feasibility target at j th decision stage
- ξ = set of uncertain (random) parameters
- ξ_i = i th member of uncertain parameter set
- $\xi^{(j)}$ = uncertain parameters realized between stages j and $j+1$
- $\hat{\xi}^{(j)}$ = realizations of j th stage uncertainty

References

- [1] Nam, T., 2007, "A Generalized Sizing Method For Revolutionary Concepts Under Probabilistic Design Constraints," Ph.D. thesis, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA.
- [2] Nam, T., and Mavris, D. N., 2008, "Multi-Stage Reliability-Based Design Optimization for Aerospace System Conceptual Design," 49th AIAA/ASME/

- ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Apr. 7–10, 2008, Schaumburg, IL.
- [3] Roedler, G. J., and Jones, C., 2005, *Technical Measurement: A Collaborative Project of PSM, INCOSE, and Industry*, Ver. 1, INCOSE-TP-2003-020-01, San Diego, CA, December 2005.
 - [4] Paulson, E. J., and Starkey, R. P., 2011, “Methodology for Variable Fidelity Multistage Optimization Under Uncertainty,” 13th AIAA Non-Deterministic Approaches Conference, Apr. 4–7, 2011, Denver, CO.
 - [5] Kreyszig, E., 1993, *Advanced Engineering Mathematics*, 7th ed., Wiley, New York, NY, pp. 344–363.
 - [6] Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P., 1996, *Numerical Recipes in Fortran 77*, 2nd ed., Cambridge University, Cambridge, MA, pp. 355–360.
 - [7] Jones, D. R., Perttunen, C. D., and Stuckman, B. E., 1993, “Lipschitzian Optimization Without the Lipschitz Constant,” *J. Optim. Theory Appl.*, **79**, pp. 157–181.